

Hard Thermal Loops and Beyond in the Finite Temperature World-Line Formulation of QED

R. Venugopalan^{1,2} and J. Wirstam²

¹*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton NY 11973.*

²*Department of Physics, Brookhaven National Laboratory, Upton, NY 11973.*

Abstract

We derive the hard thermal loop action for soft electromagnetic fields in the finite temperature world-line formulation at imaginary time, by first integrating out the hard fermion modes from the microscopic QED action. Further, using the finite T world-line method, we calculate all static higher order terms in the soft electromagnetic field. At high T , the leading non-linear terms are independent of the temperature and, except for a term quartic in the time component of the vector potential, they cancel exactly against the vacuum contribution. The remaining T -dependent non-linear terms become more strongly suppressed by the temperature as the number of soft fields increases, thus making the expansion reliable. Applications of this method to other theories and problems at the soft scale are also briefly discussed.

I. INTRODUCTION

An important feature of interacting field theories at finite temperature is the generation of thermally induced mass scales, proportional to the coupling constant g [1,2]. When an ensemble of interacting particles at temperature T is disturbed, the system responds by creating excitations at the energy scale $gT \ll T$. Here we assume $g \ll 1$. Consequently, there appears another mass scale due to interactions, the soft scale gT , in addition to the hard scale T that characterizes the average energy of the constituent particles¹. Since $(gT)^{-1}$ is much larger than the average particle separation T^{-1} , it is clear that such an excitation is a collective phenomenon of the system.

A natural question then is how to describe the physics at the soft scale in terms of the effective degrees of freedom. This problem has been studied intensively during the last decade, starting with the identification of the relevant Feynman loop diagrams that has

¹This statement is true for QED. In general, there may be additional scales that arise due to interactions. In QCD, one also has scale g^2T due to screening of magnetic interactions.

to be taken into account in amplitudes involving soft external fields [3–5]. Named Hard Thermal Loops (HTL), the diagrammatic approach then led to considerations of the low energy effective action that captures the physics at the soft scale.

Such an effective action was first considered in [6], and a very simple expression was later obtained in [7]. For example, the behavior of the soft electromagnetic field is described by the following gauge invariant effective Lagrangian,

$$L_{\text{soft}} = -\frac{m_D^2}{2} \int \frac{d\Omega}{4\pi} F_{\mu\gamma} \left(\frac{\hat{K}_\gamma \hat{K}_\sigma}{(\hat{K} \cdot \partial)^2} \right) F_{\mu\sigma} , \quad (1)$$

where $F_{\mu\gamma}$ is the field strength tensor, $\hat{K}_\mu = (i, \hat{k})$ with \hat{k} the unit three vector, ∂_μ the partial derivative and $m_D^2 = (e^2 T^2/6)$ the thermally induced Debye mass. The HTL action was later on derived in a variety of approaches, for example, from transport theory by truncating the Schwinger-Dyson equations [8] as well as by purely classical kinetic treatments [9].

In this paper, we demonstrate that the HTL effective theory can be obtained rather straightforwardly from the full microscopic action by simply imposing a momentum cut-off and integrating out the hard modes. Such a procedure is close to the renormalization group picture of Wilson (at the one-loop level), although we will neglect all cut-off dependent terms that are subleading compared to T in the resulting determinant. In this respect, our approach is similar to the derivation in [10]. However, we use a completely different method to evaluate the determinant, namely the world-line formulation, wherein the effective action is written as a quantum-mechanical path integral.

At $T = 0$, the world-line method has been developed and successfully applied to a variety of problems for a long time (see [11] and references therein). In [12–14] the formulation was extended to finite temperatures, and the finite T method was subsequently used in several studies of the effective action [15,16]. One of our main objectives here is to demonstrate that this formalism has further feasible and attractive applications at finite temperature. As we show with the example of the HTL effective action in QED, the finite T imaginary time world-line method can be useful even for calculations of non-static quantities.

In addition to the HTL effective action, we also discuss the higher order, non-linear terms. Indeed, one of the true advantages of the world-line method, compared to diagrammatic calculations, is that it comprises the sum of all 1-loop diagrams to a given order in the external field. Using the relative simplicity of the world-line method, we demonstrate how the static contribution to the effective action, for an arbitrary number of soft external photons, can be obtained. Although these terms beyond quadratic order are well known to not be of the HTL type, the non-linear interactions are needed for studies of light-light scattering and photon splitting (see [17] and references therein). Such non-linear processes have by now been observed in laboratory experiments [18], and they may also be important to explain astrophysical effects such as the observed γ -ray spectrum from very massive objects [19]. Even though the kinematical region considered here may not be encountered around, for example, neutron stars, it is nevertheless interesting to explore the form of the effective action under different circumstances and obtain a more general and complete picture. In that sense, our results reported in this paper adds rather nicely to the earlier investigations in [20–22].

At this point we should mention that we have not been able to generalize the non-static case beyond the hard thermal loop action. Including the dynamical information is clearly

of outmost importance. Hopefully the results reported here can serve as a first step in that direction.

Finally, it should also be noted that the world-line formalism provides an intuitive connection between classical kinetic theory [23] and the underlying quantum field theory. At finite T , this was noted in [24] and later explored in [25]. Remarkably, a world-line approach, analogous to that used to obtain classical kinetic theory, can be applied to derive an effective action for small- x physics [26].

The paper is organized as follows. In the next section, we recall how to go from determinants to quantum-mechanical path integrals and write down the one-loop effective action for soft gauge fields in finite temperature QED. In section III, we explicitly derive the HTL effective action from the one loop effective action. In section IV, we go beyond the HTL approximation and obtain a general expression for the effective action to all orders in the static component of the gauge field. We end in section V with our conclusions and a discussion of further applications. Details of the computations are discussed in the appendices.

II. FROM DETERMINANTS TO QUANTUM-MECHANICAL PATH INTEGRALS

The usefulness of rewriting one-loop effective actions in the language of first-quantized, quantum-mechanical path integrals was first emphasized by Strassler [28]. He showed that the method was equivalent to the Bern-Kosower rules [29] for one-loop n -point scattering amplitudes in gauge theories, the latter having been derived originally within the framework of string theory. In this section, we will begin by briefly reviewing the work of Strassler. For a comprehensive review of later developments, we refer the reader to [11]. Following [12,13], we then proceed to the finite temperature imaginary time case.

A. Scalar QED at $T = 0$

To be specific, consider as an example scalar QED in four Euclidean dimensions. The effective action Γ_{eff} for an electromagnetic background field is,

$$\Gamma_{\text{eff}} = -\log \left[\int \mathcal{D}\phi^* \mathcal{D}\phi e^{-\int d^4x (\phi^* (-D^2) \phi)} \right] , \quad (2)$$

where $D^2 = (\partial_\mu + ieA_\mu)^2$, with A_μ the electromagnetic vector field and e the electric charge. Performing the Gaussian integration, and using the Schwinger proper time method, we obtain

$$\Gamma_{\text{eff}} = \log \text{Det} [-D^2] = - \int d^4x \int_0^\infty \frac{dt}{t} \langle x | e^{-t(-D^2)} | x \rangle . \quad (3)$$

We can interpret the matrix element in Eq. (3) as a transition amplitude, and express it in the form of a quantum mechanical path integral

$$\Gamma_{\text{eff}} = - \int_0^\infty \frac{dt}{t} \int d^4x \mathcal{N} \int_{x_\mu(0)=x}^{x_\mu(t)=x} \mathcal{D}x e^{-S_{\text{w.l.}}} = - \int_0^\infty \frac{dt}{t} \mathcal{N} \int_{x_\mu(0)}^{x_\mu(t)} \mathcal{D}x e^{-S_{\text{w.l.}}} , \quad (4)$$

with the world-line action $S_{\text{w.l.}}$ given by the expression

$$S_{\text{w.l.}} = \int_0^t d\tau \left[\frac{[\dot{x}_\mu(\tau)]^2}{4} - ie\dot{x}(\tau) \cdot A(x(\tau)) \right]. \quad (5)$$

Here $\dot{x}^\mu = dx^\mu/d\tau$. The normalization constant \mathcal{N} in the effective action ensures that the free ($e = 0$) theory satisfies the condition

$$\mathcal{N} \int_{x_\mu(0)=x}^{x_\mu(t)=x} \mathcal{D}x e^{-\int_0^t d\tau (\dot{x}_\mu^2/4)} = \frac{1}{16\pi^2 t^2}. \quad (6)$$

Physically, Eq. (4) can be interpreted as the path integral for the first-quantized theory of a particle interacting with a classical background field. The endpoint positions of the particle are constrained by $x(0) = x(t) = x$, with an integration over all possible locations x and intervals $t = \Delta\tau$. It should be noted though that the variable τ does not represent any real time but is merely a parametrization of the trajectory. As such, a re-parametrization $\tau \rightarrow \tilde{\tau}(\tau)$ does not affect the classical action if the associated square-root of the one-dimensional metric, the einbein ϵ , transforms in the proper way. However, the complete path integral is not invariant unless one integrates over the space of all metrics as well [30]. In the case at hand, the einbein has been fixed, without loss of generality, to $\epsilon = 2$.

The above method can easily be generalized to include spinning particles as well. This is achieved by representing the trace over spinor indices as a path integral over anticommuting Grassmann fields in a supersymmetric generalization of Eq. (4) [28,31]. This procedure applies in general to internal symmetries, and as was shown in [32], even additional internal indices such as color can be represented by a true scalar world-line action.

B. QED at Finite Temperature

We will now derive the corresponding form of the effective action at finite T for soft external photons in QED, where the soft photons have a momenta $p \sim eT \ll T$. We will further assume that the soft momenta are much larger than the electron mass, $p \gg m$, so that the electrons can be considered massless. The microscopic, finite temperature imaginary time action is

$$S_{\text{micro}} = \int_0^\beta dx_4 \int d^3x \left[\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} + i\tilde{\psi} \gamma_\mu (\partial_\mu + ie\tilde{A}_\mu) \tilde{\psi} \right], \quad (7)$$

where $x_4 \in [0, \beta = 1/T]$ and $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$, with the tildes for later notational simplicity. As is well known in the imaginary time formalism, the gauge fields are periodic in the compact direction, $\tilde{A}_\mu(\vec{x}, x_4 + \beta) = \tilde{A}_\mu(\vec{x}, x_4)$, whereas the fermions are anti-periodic, $\tilde{\psi}(\vec{x}, x_4 + \beta) = -\tilde{\psi}(\vec{x}, x_4)$.

Since both the fermion and gauge fields contain all possible momenta, we pick a scale Λ , such that $eT \ll \Lambda \ll T$, and then split the fields into two parts; one part contains the soft momenta and the other, to be integrated out, contains the hard momenta. By performing this cut-off procedure, we introduce an explicit Λ -dependence into the effective action and also break the gauge invariance. However, the terms in the resulting effective action that depend on the scale Λ are subleading, since $\Lambda/T \ll 1$. They can therefore be ignored here

since we will only be interested in the leading dependence on T . The remaining terms, being independent of Λ , thus have to be gauge invariant.

To derive the one-loop effective action for a soft electromagnetic field one can neglect the hard photons. We therefore take $\tilde{A}_\mu = A_\mu$. Simultaneously, we write $\tilde{\psi} = \psi + \Psi$, where Ψ contains the hard momenta. We then use the background field method [33] to expand the action in Eq. (7) to quadratic order in Ψ . Integrating out the hard field, we obtain the effective action for the soft photon modes

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu}^2 + i\bar{\psi}\gamma_\mu(\partial_\mu + ieA_\mu)\psi \right) - \log \text{Det} [i\mathcal{D}] \equiv \tilde{S}_{\text{soft}} + S_{\text{soft}} , \quad (8)$$

where $\mathcal{D} = \gamma_\mu(\partial_\mu + ieA_\mu)$ and S_{soft} contains the determinant. The determinant can be re-written as

$$\text{Det} [i\mathcal{D}] = \sqrt{(\text{Det} [i\mathcal{D}])^2} = \sqrt{\text{Det} [-D^2 \mathbf{1} + ie\sigma_{\mu\nu}F_{\mu\nu}]} = \sqrt{\text{Det} [-D^2] (\mathbf{1} - ie\sigma_{\mu\nu}F_{\mu\nu}/D^2)} , \quad (9)$$

where $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/4$.

Now, ∂_μ acting on the hard field gives a contribution $\sim T$, whereas the magnitude of the soft field A_μ is $\sim T$. By power counting arguments, it follows that the term $eF_{\mu\nu}/D^2$ is sub-leading compared to the $\mathbf{1}$ in Eq. (9) when it comes to the HTL effective action. Thus the dominant correction from integrating out the hard fermion field becomes (after taking a trivial trace over spinor indices)

$$S_{\text{soft}} = -2 \text{Tr} \log [-D^2] . \quad (10)$$

Apart from an overall numerical factor, the important difference between Eq. (10) and the corresponding equation at $T = 0$, Eq. (3), stems from the fact that the finite T determinant has to respect the appropriate periodicity conditions in the compact x_4 -direction. In this case, it therefore has to be evaluated on the space of anti-periodic functions, due to the anti-periodic boundary condition of the fermion field. As was shown in ref. [13], the resulting quantum-mechanical path integral is

$$S_{\text{soft}} = 2 \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t} \mathcal{N} \int_{x_\mu(0)+n\beta\delta_{\mu 4}}^{x_\mu(t)=} \mathcal{D}x e^{-S_{\text{w.l.}}} , \quad (11)$$

where $S_{\text{w.l.}}$ is the world-line action in Eq. (5).

In the imaginary time formalism of world-lines, a finite temperature corresponds to the fact that the path can wind an arbitrary number of times around the cylinder $S^1 \times R^3$ before returning to the starting point. For $n = 0$ there is no winding, so that particular term is insensitive to the compact form of x_4 and therefore gives the $T = 0$ contribution.

III. HARD THERMAL LOOPS FROM WORLD-LINES

In this section, we will derive the HTL effective action for soft photons from Eq. (11). Many of the details of the computation are given in Appendix A. We begin by noting that Eq. (11) is the expectation value for the Wilson line, and can be expanded in powers of A_μ ,

$$S_{\text{soft}} = \left\langle e^{ie \int_0^\infty d\tau \dot{x} \cdot A} \right\rangle = \left\langle S_{\text{soft}(0)} + S_{\text{soft}(1)} + S_{\text{soft}(2)} + \dots \right\rangle, \quad (12)$$

where the subscript (i) indicates the number of external fields. The individual contributions are all gauge invariant under the periodic gauge transformations $A_\mu \rightarrow A_\mu - (1/e)\partial_\mu \omega$, where $\omega(\vec{x}, x_4 + n\beta) = \omega(\vec{x}, x_4)$.

To lowest order in the coupling constant, the path integral describes a free theory, and the corresponding contribution to S_{soft} is easily obtained. Performing the substitution

$$x_\mu(\tau) \rightarrow u_\mu(\tau) + n\beta\delta_{\mu 4}\tau/t + z_\mu, \quad (13)$$

with z_μ a τ -independent constant and $u_\mu(t) = u_\mu(0) = 0$, we use Eq. (6) to get

$$\begin{aligned} S_{\text{soft}(0)} &= 2V_4 \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t} \left(\frac{1}{16\pi^2 t^2} \right) e^{-n^2\beta^2/4t} = \frac{V_4}{8\pi^2} \int_0^\infty \frac{dt}{t^3} + \\ &\quad \frac{V_4}{4\pi^2} \sum_{n=1}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t^3} e^{-n^2\beta^2/4t}, \end{aligned} \quad (14)$$

with V_4 the four-volume. The first term is ultraviolet divergent but independent of the temperature, whereas the second term gives

$$S_{\text{soft}(0)} \Big|_{T>0} = -\frac{7\pi^2 T^4 V_4}{180}, \quad (15)$$

which is the expected result for free fermions when the infrared cutoff Λ is neglected.

The next term $S_{\text{soft}(1)}$ in Eq. (12) is linear in the soft external field and vanishes, due the sum over winding modes being odd and $\int_0^t d\tau_1 \dot{u}_\mu(\tau_1) = u_\mu(t) - u_\mu(0) = 0$. This result is of course just a manifestation of Furry's theorem.

Now consider the term quadratic in the external field, $S_{\text{soft}(2)}$. For simplicity, we will work in a gauge where A_4 is time independent, namely $\partial_4 A_4 = 0$. The spatial part A_i ($i = 1, 2, 3$) of the vector field remains nonstatic and transforms under the residual static gauge transformations as $A_i \rightarrow A_i - (1/e)\nabla_i \omega$. There are three kinds of terms to consider: (A_4-A_4) , (A_i-A_j) and (A_4-A_i) . These will be discussed below, with some of the details of the derivation given in Appendix A.

A. The A_4 - A_4 term

Expanding out the effective action to quadratic order, the most general form of the effective action is given by Eq. (A.2). For the term containing two powers of A_4 , Eq. (A.2) can be simplified to read

$$\begin{aligned} S_{\text{soft}(2)}(A_4-A_4) &= -2e^2 \int \frac{d^3 p}{(2\pi)^3} \int_0^\beta dz_4 A_4^{(1)} A_4^{(2)} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t} e^{-n^2\beta^2/4t} \mathcal{N} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \times \\ &\quad \int \mathcal{D}u \left(\dot{u}_4(\tau_1) + \frac{n\beta}{t} \right) \left(\dot{u}_4(\tau_2) + \frac{n\beta}{t} \right) \exp \left[- \int_0^t d\tau \left\{ \dot{u}^2/4 - i\vec{p} \cdot \vec{u} (\delta(\tau - \tau_2) - \delta(\tau - \tau_1)) \right\} \right], \end{aligned} \quad (16)$$

where $A_4^{(1)} = A_4(\vec{p})$ and $A_4^{(2)} = A_4(-\vec{p})$.

Following Strassler [28], we now exponentiate the vector potential under the condition that only terms linear in each $A_4^{(i)}$ contribute:

$$\left(\dot{u}_4(\tau_i) + \frac{n\beta}{t} \right) A_4^{(i)} = e^{(\dot{u}_4(\tau_i) + \frac{n\beta}{t}) A_4^{(i)}} \Big|_{\text{linear in each } A_4^{(i)}} . \quad (17)$$

As discussed in appendix A (see Eqs. (A.4)–(A.10)), this exponentiation allows one to re-write the exponential in Eq. (16) in terms of one dimensional Green's functions of a free particle constrained to a circle of circumference t . Subsequently, the path integral can be made Gaussian after a change of variables. Since the Green's function and its derivatives are known, one finds, by defining $\gamma = \vec{p}^2 x(1-x)$ and using Eqs. (A.13) and (A.14), that the finite temperature part ($n \neq 0$) of the effective action (16) becomes,

$$S_{\text{soft}(2)}(A_4 - A_4) \Big|_{T>0} = -\frac{2e^2}{\pi^2} \int \frac{d^3 p}{(2\pi)^3} A_4(\vec{p}) A_4(-\vec{p}) \int_0^\beta dz_4 \int_0^1 dx (1-x) \sum_{n=1}^\infty (-1)^n \times \\ \left(\gamma K_2(n\beta\sqrt{\gamma}) - \frac{\sqrt{\gamma}}{n\beta} K_1(n\beta\sqrt{\gamma}) [1 - \delta(x)] \right) . \quad (18)$$

The leading contribution in Eq. (18) is obtained when $\beta\sqrt{\gamma} \rightarrow 0$. In this limit, we can use

$$\lim_{\beta\sqrt{\gamma} \rightarrow 0} K_2(n\beta\sqrt{\gamma}) = \frac{2}{n^2 \beta^2 \gamma} , \quad \lim_{\beta\sqrt{\gamma} \rightarrow 0} K_1(n\beta\sqrt{\gamma}) = \frac{1}{n\beta\sqrt{\gamma}} , \quad (19)$$

since the sum over the winding modes remains convergent. Employing the identity, $\int_0^1 dx (1-x)[1 - \delta(x)] = 0$, with the usual convention [12,28] for the sign function, $\epsilon(0) = 0$, we obtain finally the result

$$S_{\text{soft}(2)}(A_4 - A_4) \Big|_{T>0} = -\frac{2e^2 T^2}{\pi^2} \int \frac{d^3 p}{(2\pi)^3} A_4(\vec{p}) A_4(-\vec{p}) \int_0^\beta dz_4 \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \\ = \frac{e^2 T^2}{6} \int \frac{d^3 p}{(2\pi)^3} A_4(\vec{p}) A_4(-\vec{p}) \int_0^\beta dz_4 = m_D^2 \int d^4 z [A_4(\vec{z})]^2 , \quad (20)$$

where $m_D^2 = e^2 T^2/6$ is the Debye mass. In the static limit, this result is well known to be the only leading contribution [1,2], and gives rise to a screening of the potential between two static test charges.

B. The A_k - A_j term

The derivation of this part of the HTL effective action is not as straightforward as the A_4 - A_4 term because the spatial components of the vector field also depend on x_4 , $A_i = A_i(\vec{x}, x_4)$. Using Eqs. (A.12) and (A.15) in Appendix A, the effective Lagrangian for this term becomes

$$L_{\text{soft}(2)}(A_k - A_j) = \frac{e^2}{8\pi^2} \sum_{n=-\infty}^\infty (-1)^n \int_0^\infty \frac{dt}{t} e^{-n^2 \beta^2 / 4t} \int_0^1 dx (1-x) e^{-ip_4 n \beta x} e^{-t P^2 x(1-x)} \\ \times \left\{ p_k p_j A_k^{(1)} A_j^{(2)} [1 - 2x]^2 + \frac{2}{t} \vec{A}^{(1)} \cdot \vec{A}^{(2)} [1 - \delta(x)] \right\} , \quad (21)$$

where the shorthand notation is $A^{(1)} = A(P)$ and $A^{(2)} = A(-P)$, with P^2 a Euclidean four-vector, $P^2 = p_4^2 + \vec{p}^2$ and $p_4 = \omega_l = 2\pi lT$ the external Matsubara frequency.

We now split the effective Lagrangian (21) into a piece $L^{(1)}$ for the term including the delta function, and another $L^{(2)}$ containing the rest of the expression. The finite T part of $L^{(1)}$ can be evaluated directly:

$$\begin{aligned} L_{\text{soft}(2)}^{(1)} \Big|_{T>0} &= \frac{-e^2}{4\pi^2} \vec{A}^{(1)} \cdot \vec{A}^{(2)} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} \frac{dt}{t^2} e^{-n^2 \beta^2 / 4t} \int_0^1 dx \delta(x) (1-x) e^{-ip_4 n \beta x - t P^2 x(1-x)} \Big|_{T>0} \\ &= \left(\frac{e^2 T^2}{12} \right) \vec{A}^{(1)} \cdot \vec{A}^{(2)} . \end{aligned} \quad (22)$$

This term looks like a magnetic mass-term, but since such a term is forbidden in QED [34] it has to be cancelled by the contributions from $L^{(2)}$. In addition, it also violates the residual gauge invariance.

As in [13], the remaining terms in $L^{(2)}$ can be simplified further by first using the substitutions $n \rightarrow -n$ and $y = 1 - x$. If $f(x, n)$ denotes the part of $L^{(2)}$ that only has to be summed over n and integrated over x to produce the Lagrangian, we have $\sum_n \int_0^1 dx (1-x) f(x, n) = \sum_n \int_0^1 dy y f(y, n)$, by using the fact that p_4 is a discrete Matsubara frequency. As in the purely static case, we can then write the result in terms of Bessel functions of the second kind, to find the finite T contribution

$$\begin{aligned} L_{\text{soft}(2)}^{(2)} \Big|_{T>0} &= \left(\frac{e^2}{4\pi^2} \right) \int_0^1 dx (1-2x) \sum_{n=1}^{\infty} (-1)^n K_0 \left(n \beta \sqrt{P^2 x(1-x)} \right) \\ &\quad \times \left\{ p_k p_j A_k^{(1)} A_j^{(2)} (1-2x) \cos[n p_4 \beta x] + 2 P^2 \vec{A}^{(1)} \cdot \vec{A}^{(2)} \int_0^x d\sigma \cos[n p_4 \beta \sigma] \right\} . \end{aligned} \quad (23)$$

To obtain the second term in the above equation, we performed an integration by parts in x , and then defined the integral over the auxiliary variable σ to re-write a sine-function as a cosine.

We now use the following identity for the infinite sum over winding modes [35],

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n \cos[n\kappa] K_0(n\phi) &= \frac{1}{2} \left[\gamma_E + \log \left(\frac{\phi}{4\pi} \right) \right] + \frac{\pi}{2} \sum_{m=0}^{\infty} \left[\frac{1}{\sqrt{\phi^2 + [(2m+1)\pi - \kappa]^2}} \right] + \\ &\quad \frac{\pi}{2} \sum_{m=0}^{\infty} \left[\frac{1}{\sqrt{\phi^2 + [(2m+1)\pi + \kappa]^2}} \right] - \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{(m+1)} , \end{aligned} \quad (24)$$

where γ_E is Euler's constant. In fact, this identity converts the sum over winding modes into a sum over fermion Matsubara frequencies. Keeping only the leading temperature dependence of Eq. (24), we show in Eqs. (A.16)–(A.19) that the finite T Lagrangian (23) becomes

$$\begin{aligned} &L_{\text{soft}(2)}^{(2)} \Big|_{T>0} \\ &= \left(\frac{e^2}{2\pi^2} \right) A_k^{(1)} A_j^{(2)} \frac{\partial^2}{\partial p_k \partial p_j} T \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\theta \theta^2 \int_0^1 \frac{dx}{x(1-x)} \log \left[P^2 x(1-x) + \theta^2 + (k_4 - x p_4)^2 \right] , \end{aligned} \quad (25)$$

where $k_4 = (2m+1)\pi T$ plays the role of the fermion Matsubara frequency.

Now, to extract the HTL effective action from Eq. (25) we note that the leading contribution is equivalent to a hard thermal loop integral, multiplied by $A_k^{(1)}A_j^{(2)}$. This can be seen as follows. First take the derivatives with respect to p_k and p_j to get

$$L_{\text{soft}(2)}^{(2)}|_{T>0} = \left(\frac{e^2}{\pi^2}\right) A_k^{(1)} A_j^{(2)} \left[T \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\theta \theta^2 \int_0^1 dx \left\{ \frac{\delta_{kj}}{P^2 x(1-x) + \theta^2 + (k_4 - xp_4)^2} - \frac{2p_j p_k x(1-x)}{[P^2 x(1-x) + \theta^2 + (k_4 - xp_4)^2]^2} \right\} \right]. \quad (26)$$

Partially integrating over θ in the first term in Eq. (26), and neglecting a temperature independent UV-divergent constant, we are left with

$$L_{\text{soft}(2)}^{(2)}|_{T>0} = \left(\frac{2e^2}{3\pi^2}\right) A_k^{(1)} A_j^{(2)} T \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\theta \theta^2 \int_0^1 dx \left(\frac{\theta^2 \delta_{kj} - 3p_j p_k x(1-x)}{[P^2 x(1-x) + \theta^2 + (k_4 - xp_4)^2]^2} \right). \quad (27)$$

Since the θ -dependent integrand only contains multiples of θ^2 , we can formally interpret θ as the magnitude of a three-vector, so that $\theta^2 \rightarrow |\vec{\theta}|^2$ and $d\theta \theta^2 \rightarrow d^3\theta/(4\pi)$. By adding the terms $A_k^{(1)}A_j^{(2)}[x(\theta_k p_j + \theta_j p_k) - \theta_k p_j]$, which are all odd in $\vec{\theta}$, and changing the integration variable $\vec{\theta} = \vec{k} - x\vec{p}$, we obtain the leading contribution,

$$L_{\text{soft}(2)}^{(2)}|_{T>0} = 4e^2 A_k^{(1)} A_j^{(2)} T \sum_{m=-\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \frac{k_k k_j}{K^2 (K - P)^2}, \quad (28)$$

where $K^2 = k_4^2 + \vec{k}^2$. Once the sum has been performed, one can continue p_4 to arbitrary Euclidean values, so that it becomes the analytical continuation of the energy in Minkowski space-time. When both $p_4 \ll T$ and $|\vec{p}| \ll T$, as is the case when the fields are soft, the integral in Eq. (28) is well known [2]. The leading contribution from Eq. (25) is

$$L_{\text{soft}(2)}^{(2)}|_{T>0} = -\frac{e^2 T^2}{12} \vec{A}(P) \cdot \vec{A}(-P) + \frac{e^2 T^2}{6} \int \frac{d\Omega}{4\pi} \hat{k} \cdot \vec{A}(P) \left[\frac{ip_4}{\hat{K} \cdot P} \right] \hat{k} \cdot \vec{A}(-P). \quad (29)$$

Thus, as anticipated, the first term in (29) cancels the result in Eq. (22). The total effective Lagrangian for the spatial part of the electromagnetic field is then

$$L_{\text{soft}(2)}(A_k - A_j)|_{T>0} = \frac{e^2 T^2}{6} \int \frac{d\Omega}{4\pi} \hat{k} \cdot \vec{A}(P) \left[\frac{ip_4}{\hat{K} \cdot P} \right] \hat{k} \cdot \vec{A}(-P). \quad (30)$$

This term is gauge invariant under the residual static gauge transformations allowed by the condition $\partial_4 A_4 = 0$.

C. The A_4 - A_k term

Finally, we have to consider the part of the effective action that contains one spatial component A_k together with one power of A_4 . Since the A_4 -field is static, we have after a Fourier transformation (see Eq. (A.2))

$$S_{\text{soft}(2)}(A_4 - A_k) = -2e^2 \int \frac{d^3p}{(2\pi)^3} \sum_n (-1)^n \int_0^\infty \frac{dt}{t} e^{-n^2\beta^2/4t} \mathcal{N} \int \mathcal{D}u e^{-\int_0^t d\tau \dot{u}^2/4} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \times$$

$$\left(\dot{u}_4(\tau_1) + \frac{n\beta}{t} \right) \dot{u}_k(\tau_2) A_4(\vec{p}) A_k(-\vec{p}) e^{i\vec{p} \cdot (\vec{u}(\tau_1) - \vec{u}(\tau_2))} . \quad (31)$$

However, the $n\beta/t$ -term vanishes since the sum is odd in n , whereas the \dot{u}_4 -term gives, after an exponentiation in the same way as before,

$$S_{\text{soft}(2)}(A_4 - A_k) =$$

$$-\frac{e^2}{8\pi^2} \int \frac{d^3p}{(2\pi)^3} \sum_n (-1)^n \int_0^\infty \frac{dt}{t^3} e^{-n^2\beta^2/4t} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-\vec{p}^2 G_B(\tau_1, \tau_2) - i\vec{p} \cdot \vec{A} \partial_{\tau_2} G_B(\tau_1, \tau_2)} \Bigg|_{\substack{\text{linear in} \\ A_k \text{ and } A_4}} . \quad (32)$$

But this part of the effective action has no dependence on A_4 . Therefore, the exponent cannot be expanded to give any term linear in A_4 , and the Lagrangian vanishes to all orders in p/T .

D. Summing the contributions

The complete effective action to quadratic order in the soft photons then becomes, by adding the contributions from Eqs. (20) and (30)

$$S_{\text{soft}} = m_D^2 \int d^4x \left([A_4(\vec{x})]^2 + \int \frac{d\Omega}{4\pi} \hat{k} \cdot \vec{A}(x) \left\{ \frac{i\partial_4}{\vec{K} \cdot \partial} \right\} \hat{k} \cdot \vec{A}(x) \right) . \quad (33)$$

We have thus derived the effective action by first integrating out the hard modes from the microscopic action, and then evaluating the determinant form of the one-loop effective action in Eq. (8) as a quantum-mechanical path integral. The general HTL action in Eq. (1) of course reduces to the above result when the gauge condition $\partial_4 A_4 = 0$ is imposed.

IV. BEYOND HARD THERMAL LOOPS

In this section, we turn to the higher order terms in the effective action. Although these terms do not have HTL's, it is nevertheless interesting to study the form of the effective action in the high T phase, and obtain a systematic expansion in powers of p/T . As mentioned already in the introduction, the non-linear effective action is a useful tool in astrophysical applications [20]. Moreover, one could use the resulting non-linear expression to contract and thermalize any pair of external legs, and by this procedure obtain higher corrections to, for example, the pressure.

The power of the world-line approach becomes manifest as one goes to higher orders. For instance, to obtain the dominant term at high temperature, one should note that individual loops in the diagrammatic approach may contain superficially leading terms that only cancel when all contributing graphs are summed. In marked contrast, within the world-line formalism, all diagrams are accounted for in the effective action.

In [21,22] a diagrammatic approach was used to argue that all terms with $N \geq 4$ external photons have a finite limit when $T \rightarrow \infty$. In the world-line formulation, we will verify explicitly that this is true for the static terms. In principle², we can also obtain all corrections in a power expansion of p/T . For $N = 4$, parts of our results overlap with the world-line calculation in [16].

Proceeding in the same way as before, by generalizing straightforwardly Eq. (A.12) of Appendix A to the N 'th order, we have in the static case

$$S_{\text{soft}}(N) = \frac{(-ie)^N}{8\pi^2} \int_0^\beta dx_4 \int \frac{d^3 p_1}{(2\pi)^3} \cdots \frac{d^3 p_N}{(2\pi)^3} \delta^3(p_1 + \dots + p_N) \sum_n (-1)^n \int_0^\infty dt t^{N-3} e^{-n^2 \beta^2 / 4t} \times$$

$$\exp \left[\frac{n\beta}{t} \sum_{i=1}^N A_4^{(i)} \right] \int_0^1 dx_1 \cdots \int_0^{x_{N-1}} dx_N \exp \left[- \sum_{\substack{i,j=1 \\ i < j}}^N \{ t \vec{p}_i \cdot \vec{p}_j (x_i - x_j) (1 - (x_i - x_j)) \} \right.$$

$$\left. - \sum_{\substack{i,j=1 \\ i < j}}^N \left\{ i(\vec{p}_i \cdot \vec{A}^{(j)} - \vec{p}_j \cdot \vec{A}^{(i)}) (1 - 2(x_i - x_j)) + \frac{2}{t} A^{(i)} \cdot A^{(j)} (1 - \delta(x_i - x_j)) \right\} \right] , \quad (34)$$

where $A^{(i)} = A(p_i)$, and the only term in the field-dependent exponential to be kept is the one linear in each $A^{(i)}$. Since the effective action reflects the symmetries of the original microscopic action, and in particular the invariance under charge conjugation, all odd powers of the vector field vanish and the only contribution comes from N even. This generalizes the explicit calculation for a single external field discussed in section III.

We now expand the exponential $\exp(n\beta \sum_{i=1}^N A_4^{(i)}/t)$ in powers of A_4 . For odd powers, the sum over n is odd and vanishes. Hence we only have to consider even powers. The effective action could still contain an arbitrary even number k ($0 \leq k \leq N$) of powers of A_4 , though. Following the general procedure outlined in appendix A, we then have

$$S_{\text{soft}}(N) = \frac{(-ie)^N}{8\pi^2} \int_0^\beta dx_4 \int \frac{d^3 p_1}{(2\pi)^3} \cdots \frac{d^3 p_N}{(2\pi)^3} \delta^3(p_1 + \dots + p_N) \int_0^1 dx_1 \cdots \int_0^{x_{N-1}} dx_N \sum_{\substack{k=0 \\ k \text{ even}}}^N I_N^{(k)} , \quad (35)$$

where the function $I_N^{(k)}$ consists of three different terms,

$$I_N^{(k)} = I_{N(1)}^{(k)} + I_{N(2)}^{(k)} + I_{N(3)}^{(k)} = \sum_{n=-\infty}^{\infty} (-1)^n \left[I_{N(1)}^{(k,n)} + I_{N(2)}^{(k,n)} + I_{N(3)}^{(k,n)} \right] \quad (36)$$

The three terms in Eq. (36) correspond to the different ways of expanding the exponential in the last line of Eq. (34) to get N external fields in total, given that the term $\exp(n\beta \sum_{i=1}^N A_4^{(i)}/t)$ is expanded to an arbitrary even order k . When only the first p -dependent term of the last exponential in Eq. (34) contributes with the remaining $(N - k)$ powers of the vector field, one finds,

²With the operator D^2 , this is true for scalar QED whence the boundary conditions in x_4 is changed. In QED, one would also have to take into account the $F_{\mu\nu}$ -term to get the correct factor.

$$I_{N(1)}^{(k,n)} = \tilde{p}^{(2+k-N)} (n\beta)^{N-2} K_{2+k-N}(n\beta\tilde{p}) f_1(\{x\}) \mathcal{P}[(p_{i_{k+1}} \cdots p_{i_N})(A_{4_1} \cdots A_{4_k})(A_{i_{k+1}} \cdots A_{i_N})], \quad (37)$$

and if the $(N-k)$ powers only come from the p -independent term of the exponential, the result is

$$I_{N(2)}^{(k,n)} = \tilde{p}^{2-(N-k)/2} (n\beta)^{(N+k)/2-2} K_{2+(k-N)/2}(n\beta\tilde{p}) f_2(\{x\}) \quad (38)$$

$$\times \mathcal{P}[(A_{4_1} \cdots A_{4_k})[(A^{(k+1)} \cdot A^{(k+2)}) \cdots (A^{(N-1)} \cdot A^{(N)})]]. \quad (39)$$

Finally, there is also the possibility to expand both terms, giving

$$I_{N(3)}^{(k,n)} = \theta[(N-4) - k] \sum_{l=1}^{(N-k)/2-1} \tilde{p}^{(2+k+l-N)} (n\beta)^{N-l-2} K_{2+k+l-N}(n\beta\tilde{p}) f_3^{(l)}(\{x\}) \times \mathcal{P}[(p_{i_{k+2l+1}} \cdots p_{i_N})(A_{4_1} \cdots A_{4_k})(A_{i_{k+2l+1}} \cdots A_{i_N})[(A^{(k+1)} \cdot A^{(k+2)}) \cdots (A^{(k+2l-1)} \cdot A^{(k+2l)})]] , \quad (40)$$

where the θ -function ensures that the term vanishes when $k > (N-4)$. In all the equations above we used

$$\tilde{p}^2 = \sum_{\substack{i,j=1 \\ i < j}}^N \{\vec{p}_i \cdot \vec{p}_j (x_i - x_j)(1 - (x_i - x_j))\} , \quad (41)$$

and the functions $f_i(\{x\})$ comprise the dependence on the dimensionless variables x_i , together with some irrelevant combinatorial factors. \mathcal{P} denotes both all permutations of choosing k factors of A_4 and the other $(N-k)$ factors out of the N total, as well as all possible ways of contracting the different vector fields with themselves and/or the momentum.

To study the high temperature behavior of the effective action, we isolate the dependence on T and p in $I_{N(1)}^{(k)}$. From Eqs. (37)-(40) we find that all three terms give different contributions. As shown in appendix B, the p and T dependence of $I_{N(1)}^{(k)}$ becomes

$$I_{N(1)}^{(k)} \propto f_1(\{x\})(A_{4_1} \cdots A_{4_k})(A_{i_{k+1}} \cdots A_{i_N}) \times \left(\frac{1}{\tilde{p}} \right)^{N-4} \left[\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} z \Gamma(z) (1 - 2^{1-z}) \zeta(z) (-z-1) \cdots (-z-N+3) (\beta\tilde{p})^{-z} \times \left\{ \delta_{kN} \left[\frac{2\Gamma[(N-4+z)/2]}{\Gamma[(N+z-3)/2]} - \frac{\Gamma[(N+z-2)/2]}{\Gamma[(N+z-1)/2]} \right] + \delta_{k(N-2)} p_{i_{N-1}} p_{i_N} \tilde{p}^{-2} \frac{\Gamma[(N+z-2)/2]}{\Gamma[(N+z-1)/2]} + \sum_{\substack{j=0 \\ j \text{ even}}}^{N-4} 2^{N-j-3} \delta_{kj} p_{i_{j+1}} \cdots p_{i_N} \tilde{p}^{j-N} \left[\frac{\Gamma[(j+z)/2]}{\Gamma[(j+z+1)/2]} + \sum_{m=1}^{(N-j-2)/2} a_m^{(1)} \frac{\Gamma[(j+z+2m)/2]}{\Gamma[(j+z+2m+1)/2]} \right] \right\} \right], \quad (42)$$

where $\zeta(z)$ is the Riemann zeta-function, $a_m^{(1)}$ contains some irrelevant combinatorial factors and the sum over m runs up to the largest integer $\leq (N-j-2)/2$. Similarly, one can show that the expression for the second term $I_{N(2)}^{(k)}$, valid for all k if $N \geq 6$ and for $k \geq 2$ when $N = 4$, can be written as

$$\begin{aligned}
I_{N(2)}^{(k)} &\propto f_2(\{x\})(A_{4_1} \cdots A_{4_k})[(A^{(k+1)} \cdot A^{(k+2)}) \cdots (A^{(N-1)} \cdot A^{(N)})] \times \\
&\left(\frac{1}{\tilde{p}}\right)^{N-4} \left[\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} z \Gamma(z) (1 - 2^{1-z}) \zeta(z) (-z-1) \cdots (-z - (N+k)/2 + 3) (\beta \tilde{p})^{-z} \times \right. \\
&\left\{ \delta_{kN} \left[\frac{2\Gamma[(N-4+z)/2]}{\Gamma[(N+z-3)/2]} - \frac{\Gamma[(N+z-2)/2]}{\Gamma[(N+z-1)/2]} \right] + \delta_{k(N-2)} \frac{\Gamma[(N+z-4)/2]}{\Gamma[(N+z-3)/2]} + \right. \\
&\left. \sum_{\substack{j=0 \\ j \text{ even}}}^{N-4} 2^{(N-j)/2-3} \delta_{kj} \left[\frac{\Gamma[(j+z)/2]}{\Gamma[(j+z+1)/2]} + \sum_{m=1}^{((N-j)/2-2)/2} a_m^{(2)} \frac{\Gamma[(j+z+2m)/2]}{\Gamma[(j+z+2m+1)/2]} \right] \right\} \Bigg] . \quad (43)
\end{aligned}$$

When $N = 4$ and $k = 0$ it is easier to sum the Bessel function $K_0(n\beta\tilde{p})$ in Eq. (39) directly. The leading high temperature term contains a logarithmic dependence on T , but when integrated over x_i this contribution vanishes. This result, with four external fields, agrees with the result in [16]. Finally, the last term $I_{N(3)}^{(k)}$ becomes, for a given $k \leq N - 4$,

$$\begin{aligned}
I_{N(3)}^{(k)} &\propto \sum_{l=1}^{(N-k)/2-1} f_3^{(l)}(\{x\})(A_{4_1} \cdots A_{4_k})(A_{i_{k+2l+1}} \cdots A_{i_N})[(A^{(k+1)} \cdot A^{(k+2)}) \cdots (A^{(k+2l-1)} \cdot A^{(k+2l)})] \times \\
&\left(\frac{1}{\tilde{p}}\right)^{2(N-l)-4-k} \left[\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} z \Gamma(z) (1 - 2^{1-z}) \zeta(z) (-z-1) \cdots (-z - (N-l) + 3) (\beta \tilde{p})^{-z} \times \right. \\
&\left\{ 2^{N-k-l-3} \frac{\Gamma[(k+z)/2]}{\Gamma[(k+z+1)/2]} + \sum_{m=1}^{(N-k-l-2)/2} a_m^{(3)} \frac{\Gamma[(k+z+2m)/2]}{\Gamma[(k+z+2m+1)/2]} \right\} \Bigg] . \quad (44)
\end{aligned}$$

Even though the Eqs. (42)-(44) look complicated, the analytical structure of the integrals allow some simple but powerful conclusions. This is due to the fact that the temperature dependence is only contained in the term $(\beta\tilde{p})^{-z}$, and in each of the terms in Eqs. (42)-(44) the contour can be closed in the left side of the complex z -plane and evaluated by the residue theorem. The analytical structure of singularities in the complex z -plane therefore determines the temperature dependence of $S_{\text{soft}(N)}$.

The common function $g(z) = z\Gamma(z)(1 - 2^{1-z})\zeta(z)$ has simple poles at $z = -(2n+1)$, with n any non-negative integer. Further, the other Γ -functions that enter the expressions always appear in specific ratios, so that the zeros of the denominators in these ratios exactly cancel the poles of $g(z)$ when the Γ -function in the numerator starts to have poles along the negative real axis. This holds irrespective of the additional factors $[(-z-1)(-z-2)\cdots]$ in the numerators that actually cancel several of the poles. Consequently, there are only simple poles to evaluate. If there were any higher order poles, the factor $(\beta\tilde{p})^{-z}$ would contribute with a logarithmic factor, but since this is not the case, the effective action contains only powers of p/T .

Although the above equations give the complete, general formula, a simpler result can be used for the terms independent of the external momenta. This limit corresponds to taking the vector field to be constant, and we then have for the effective action,

$$S = 2 \sum_n (-1)^n \int_0^\infty \frac{dt}{t} \mathcal{N} \int_{\substack{x_\mu(t)= \\ x_\mu(0)+n\beta\delta_{\mu 4}}} \mathcal{D}x e^{-\int_0^t d\tau \dot{x}^2/4} e^{ieA_\mu \int_0^t d\tau \dot{x}_\mu} = \frac{2T^4 V_4}{\pi^2} \sum_n \frac{(-1)^n}{n^4} e^{ien\beta A_4} , \quad (45)$$

giving for the finite T effective potential $V(A_4)$,

$$V(A_4) = \frac{4T^4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \cos(en\beta A_4) . \quad (46)$$

Thus the potential is periodic in A_4 , with period $2\pi T/e$, as was first noted in [37]. Restricting A_4 to the interval $[-\pi T/e, \pi T/e]$ and writing $A_4 = \pi T q/e$, $-1 \leq q \leq 1$, we have

$$V(A_4) = -\frac{2T^4\pi^2}{3} \left(\frac{1}{45} - \frac{1}{24} [1 - (q_m - 1)^2]^2 \right) , \quad (47)$$

where $q_m = (q + 1)_{\text{mod}2}$ (see [36] for a discussion on this effective potential). Expanding the potential gives,

$$V(A_4) = -\frac{7\pi^2 T^4}{180} + m_D^2 (A_4)^2 - \frac{e^4}{12\pi^2} (A_4)^4 . \quad (48)$$

This result agrees with the dimensionally reduced theory in [27]³, once the variables are properly rescaled to the canonical mass dimensions in 4d. The result for $V(A_4)$ can of course be obtained from Eqs. (42)-(44) as well, and we have checked that this is indeed the case. More importantly though, they also give all the additional derivative terms.

Apart from the quartic term in Eq. (48), there is an infinite sequence, with different N , of leading terms that are independent of the temperature when $p/T \rightarrow 0$. From Eqs. (42)-(44) the temperature independent terms must come from the poles at $z = 0$, corresponding to the $k = 0$ terms. Since the $k = 0$ terms are the only ones that also appear at $T = 0$, it is interesting to compare the two contributions. We find from computing the $T = 0$ contribution of Eq. (35), arising from the $n = 0$ term in Eq. (36), that all T -independent terms in Eqs. (42)-(44) are completely cancelled.

It is straightforward to also include the $F_{\mu\nu}$ term from Eq. (9) into this calculation. In the path integral formalism [13], this term will augment the world-line form of the effective action by a factor proportional to

$$\text{Tr} \exp \left[2 \sum_{i=1}^N \sigma_{k\nu} p_{k_i} A_{\nu}^{(i)} \right] , \quad (49)$$

where $k = 1, 2, 3$ and the trace is over spinor indices. Now, this term occurs with the same powers of the variables A_{μ} , p , t and n as the second term in the last exponential of Eq. (34), namely the term

$$\exp \left[\sum_{\substack{i,j=1 \\ i < j}}^N \left\{ i(\vec{p}_i \cdot \vec{A}^{(j)} - \vec{p}_j \cdot \vec{A}^{(i)})(1 - 2(x_i - x_j)) \right\} \right] . \quad (50)$$

The structural dependence on these variables is the crucial ingredient to prove that a cancellation indeed occurs. Since we know that all the T -independent terms in Eq. (34) are

³For QCD, results for the one loop static potential were first derived by Nadkarni [38].

cancelled against the $T = 0$ effective action, the identical structures of Eq. (49) and Eq. (50) ensure that this holds true also for the $F_{\mu\nu}$ part. Consequently, apart from the quartic term in Eq. (48), all other higher order terms in the effective action for soft photons in QED are suppressed by factors of T .

As can be seen from Eqs. (42)-(44), the polynomial in $[(-z-1)(-z-2)\cdots]$ cancels all of the poles up to order N , with the exception of the pole at $z = 0$. Naively, one could imagine that the high temperature expansion would yield a series like $(1 + \beta^2 p^2 + \beta^4 p^4 + \dots)$, where the first term corresponds to the leading temperature independent part and p^2 denotes some generic combination of the external momenta. Instead, the leading, temperature dependent, contribution for N external fields is suppressed by β^a , where a is of the order of N . As a result, the terms in the effective action tend to be more strongly suppressed by powers of the inverse temperature as the number of external fields, N , increases. Since the T -independent terms by dimensional arguments have the schematic form $\partial^{(4-N)}(eA)^N$, they are all of the same order for the soft fields, where $\partial \sim eT$ and $A \sim T$: $\partial^{(4-N)}(eA)^N \sim (eT)^{(4-N)}(eT)^N \sim (eT)^4$. However, the surviving terms that are suppressed by powers of the inverse temperature become smaller in magnitude as the number of the soft, external fields increases. The expansion in powers of the soft field is therefore reliable, with the higher orders becoming more and more negligible.

In [21] a cancellation of all non-linear terms was verified in the long wave-length limit, $|\vec{p}| \rightarrow 0$, $p_0 \neq 0$. For slowly varying fields, a similar result was found by expanding in powers of m/T [20], m being the electron mass. We arrive at the same conclusion in the static ultrarelativistic limit, apart from the appearance of an $(A_4)^4$ -term, by a direct calculation of all $N \geq 4$ terms. This calculation complements the earlier ones and strongly suggests that the T independent non-linear terms beyond quartic order always cancel out, regardless of the momenta of the external fields. Whether or not the remaining terms are more strongly suppressed by the temperature, as in the static case, is not clear at the moment, although we expect this to be the case due to gauge invariance.

V. CONCLUSIONS

In this paper, we derived the hard thermal loop effective action for a soft electromagnetic field directly from the microscopic, one loop effective action in QED. This result was obtained by first integrating out the hard fermions from the original microscopic action, and subsequently computing the resulting determinant in the quantum mechanical world-line formalism. In addition, we have shown how all leading higher order static terms can be obtained with the same method. When $T \rightarrow \infty$, all the remaining terms, with the sole exception of the quartic term in the A_4 -potential, are cancelled by the $T = 0$ terms in the effective action. The remaining terms are generally suppressed by a factor $\sim \beta^a$, where a is of the order of the number of external fields. We would now like to comment on our results and speculate about some further applications wherein the methods described in this paper may prove useful.

First of all, we have neglected all dependence on the momentum cut-off, Λ . This is reasonable as long as we are only interested in the leading behavior. In general, the effective action should of course also depend on the cut-off that separates the hard modes from the soft ones. A simple momentum cut-off does not respect gauge invariance but, in principle, one

could derive the effective action in a gauge where there are only physical degrees of freedom present. In this case, the momentum cut-off does not pose any problems and one would have a true Wilson effective action $S(T, \Lambda)$ for the soft electromagnetic fields in the QED plasma, albeit valid only in that physical gauge. Such a formulation of the HTL effective action could be advantageous both for non-static quantities as well as for computations of the pressure and entropy in an HTL resummation scheme [39,40]. Furthermore, for a quantity like the pressure, even the static higher order terms can be used to obtain sub-leading contributions [41,42].

An interesting point to consider further is the connection to the random phase approximation (RPA). The computation of the one-loop effective action implies that the corresponding Green's function is obtained in an RPA—in the diagrammatic picture this corresponds to resumming a chain of one-loop fermion bubbles. This corresponds precisely to the derivation of the dressed propagator in the hard thermal loop approximation scheme. Thus, in our derivation here, we take the opposite approach to the original HTL derivation; we first calculate the effective action and from that result deduce the Green's function. The resemblance between the RPA and hard thermal loop approximation also suggests a natural direction for improving the effective theory further, and hopefully the world-line method will prove to be useful in this case as well.

Another interesting extension would be to include also the effects of a finite density in our results. This would first of all generalize the derivation of the HTL effective action. Secondly, it would be important to study to what extent the non-linear terms cancel also at finite densities, since this situation is more likely to resemble situations of astrophysical interest.

Finally, a future topic is how to include the non-static higher order terms. Even though this seems to be a formidable task in general, obtaining just the next order correction would be of great interest. In particular, a non-Abelian generalization could provide some insights into the expansion of the Wilson line in real time [43,44]. Such a continuation is not merely of academic interest, but could in fact have important consequences for the observables of the heavy ion collisions at RHIC [45].

Acknowledgments

We thank F. Gelis and R. D. Pisarski for useful discussions and for reading the manuscript. We also thank L. McLerran for emphasizing the physical context of this work. R. V. was supported by RIKEN-BNL and by BNL under DOE grant DE-AC02-98CH10886. The work of J.W. was supported by The Swedish Foundation for International Cooperation in Research and Higher Education (STINT) under contract no 99/665.

APPENDIX A: SOME USEFUL IDENTITIES

In this appendix we give some of the intermediate stages in the calculation of the hard thermal loop effective action. With the Fourier expansion

$$A_\mu(\vec{x}, x_4) = T \sum_m \int \frac{d^3p}{(2\pi)^3} A_\mu(\omega_m, \vec{p}) e^{-i(\omega_m x_4 - \vec{p} \cdot \vec{x})} , \quad (\text{A.1})$$

where $\omega_m = 2m\pi T$ is the Matsubara frequency, and using the change of variables in Eq. (13), we find the quadratic term in Eq. (12) to be, before any gauge-fixing,

$$\begin{aligned}
S_{\text{soft}(2)} &= -2e^2 T \sum_m T \sum_l \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} A_\mu(\omega_m, \vec{p}) A_\nu(\omega_l, \vec{q}) \int_0^\beta dz_4 \int d^3 z \sum_{n=-\infty}^\infty (-1)^n \times \\
&\int_0^\infty \frac{dt}{t} e^{-n^2 \beta^2 / 4t} \mathcal{N} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int \mathcal{D}u \left(\dot{u}_\mu(\tau_1) + \frac{n\beta\delta_{\mu 4}}{t} \right) \left(\dot{u}_\nu(\tau_2) + \frac{n\beta\delta_{\nu 4}}{t} \right) e^{-\int_0^t d\tau (\dot{u}^2/4)} \times \\
&\exp \left\{ i\vec{p} \cdot [\vec{u}(\tau_1) + \vec{z}] + i\vec{q} \cdot [\vec{u}(\tau_2) + \vec{z}] - i\omega_m \left[u_4(\tau_1) + z_4 + \frac{n\beta\tau_1}{t} \right] - i\omega_l \left[u_4(\tau_2) + z_4 + \frac{n\beta\tau_2}{t} \right] \right\} \\
&= -2e^2 T \sum_m \int \frac{d^3 p}{(2\pi)^3} A_\mu(\omega_m, \vec{p}) A_\nu(-\omega_m, -\vec{p}) \sum_{n=-\infty}^\infty (-1)^n \int_0^\infty \frac{dt}{t} e^{-n^2 \beta^2 / 4t} \times \\
&\mathcal{N} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int \mathcal{D}u \left(\dot{u}_\mu(\tau_1) + \frac{n\beta\delta_{\mu 4}}{t} \right) \left(\dot{u}_\nu(\tau_2) + \frac{n\beta\delta_{\nu 4}}{t} \right) e^{-\int_0^t d\tau (\dot{u}^2/4)} \times \\
&\exp \left\{ -i\omega_m [u_4(\tau_1) - u_4(\tau_2)] + i\vec{p} \cdot [\vec{u}(\tau_1) - \vec{u}(\tau_2)] - i\omega_m n\beta \frac{(\tau_1 - \tau_2)}{t} \right\} . \tag{A.2}
\end{aligned}$$

We now use the trivial identity $u(\tau_i) = \int d\tau [u(\tau)\delta(\tau - \tau_i)]$, as well as the generalization of Eq. (17),

$$\left(\dot{u}_\mu(\tau_i) + \frac{n\delta_{\mu 4}\beta}{t} \right) A_\mu^{(i)} = e^{[i\dot{u}_\mu(\tau_i) + \frac{n\delta_{\mu 4}\beta}{t}]A_\mu^{(i)}} \Big|_{\text{linear in each } A_\mu^{(i)}} , \tag{A.3}$$

with the notation $A^{(1)} = A(\omega_m, \vec{p})$, $A^{(2)} = A(-\omega_m, -\vec{p})$, to write Eq. (A.2) as

$$\begin{aligned}
S_{\text{soft}(2)} &= -2e^2 T \sum_m \int \frac{d^3 p}{(2\pi)^3} \sum_{n=-\infty}^\infty (-1)^n \int_0^\infty \frac{dt}{t} e^{-n^2 \beta^2 / 4t} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \times \\
&\exp \left[\frac{n\beta(A_4^{(1)} + A_4^{(2)})}{t} - i \frac{\omega_m n\beta(\tau_1 - \tau_2)}{t} \right] \mathcal{N} \int \mathcal{D}u \exp \left[-\int_0^t d\tau \left(\frac{\dot{u}^2}{4} + J_\mu u_\mu \right) \right] \Big|_{\text{linear in each } A_\mu^{(i)}} , \tag{A.4}
\end{aligned}$$

where the source term $J_\mu = (J_k, J_4)$ is given by

$$J_k(\tau) = -ip_k [\delta(\tau - \tau_1) - \delta(\tau - \tau_2)] - \sum_{l=1}^2 A_k^{(l)} \delta(\tau - \tau_k) \partial_{\tau_k} , \tag{A.5}$$

$$J_4(\tau) = i\omega_n [\delta(\tau - \tau_1) - \delta(\tau - \tau_2)] - \sum_{l=1}^2 A_4^{(l)} \delta(\tau - \tau_k) \partial_{\tau_k} . \tag{A.6}$$

We can then change the variable of integration,

$$\tilde{u}_\mu = u_\mu - 2 \int_0^t d\tilde{\tau} G_B(\tau, \tilde{\tau}) J_\mu(\tilde{\tau}) , \tag{A.7}$$

where $G_B(\tau_1, \tau_2)$ is the one-dimensional Green's function on a circle of circumference t ,

$$G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{t} , \tag{A.8}$$

and

$$\partial_{\tau_1} G_B(\tau_1, \tau_2) = \epsilon(\tau_1 - \tau_2) - \frac{2(\tau_1 - \tau_2)}{t} \quad , \quad \partial_{\tau_1} \partial_{\tau_2} G_B(\tau_1, \tau_2) = 2 \left[\frac{1}{t} - \delta(\tau_1 - \tau_2) \right] \quad . \quad (\text{A.9})$$

The resulting path integral then becomes Gaussian, and with the normalization in Eq. (6) we obtain (with the implicit condition that only terms containing one power of each $A^{(i)}$ is to be kept),

$$\begin{aligned} S_{\text{soft}(2)} &= -\frac{e^2}{8\pi^2} T \sum_m \int \frac{d^3 p}{(2\pi)^3} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t^3} e^{-n^2 \beta^2 / 4t} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \times \\ &\exp \left[-i \frac{\omega_m n \beta (\tau_1 - \tau_2)}{t} + \frac{n \beta (A_4^{(1)} + A_4^{(2)})}{t} \right] \exp \left[- \int_0^t d\tau \int_0^\tau d\tilde{\tau} J_\mu(\tau) G_B(\tau, \tilde{\tau}) J_\mu(\tilde{\tau}) \right] \\ &= -\frac{e^2}{8\pi^2} T \sum_m \int \frac{d^3 p}{(2\pi)^3} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t^3} e^{-n^2 \beta^2 / 4t} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \times \\ &\exp \left[-i \frac{p_4 n \beta (\tau_1 - \tau_2)}{t} + \frac{n \beta (A_4^{(1)} + A_4^{(2)})}{t} \right] \exp \left[-P^2 G_B(\tau_1, \tau_2) + i \partial_{\tau_1} G_B(\tau_1, \tau_2) \times \right. \\ &\quad \left. \left\{ p_k (A_k^{(1)} + A_k^{(2)}) - p_4 (A_4^{(1)} + A_4^{(2)}) \right\} - A_\mu^{(1)} A_\mu^{(2)} \partial_{\tau_1} \partial_{\tau_2} G_B(\tau_1, \tau_2) \right] \quad , \end{aligned} \quad (\text{A.10})$$

where we have put $p_4 = \omega_m$ and $P^2 = \vec{p}^2 + p_4^2$ in the last expression. By finally using the identity

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 f(\tau_1 - \tau_2) = t^2 \int_0^1 dx (1-x) f(xt) \quad , \quad (\text{A.11})$$

we can write Eq. (A.10) as

$$\begin{aligned} S_{\text{soft}(2)} &= -\frac{e^2}{8\pi^2} T \sum_m \int \frac{d^3 p}{(2\pi)^3} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty \frac{dt}{t} \int_0^1 dx (1-x) e^{-t P^2 x(1-x) - n^2 \beta^2 / 4t} \times \\ &\exp \left[\frac{n \beta (A_4^{(1)} + A_4^{(2)})}{t} - i p_4 n \beta x + i \epsilon_{\mu 4} \left\{ p_\mu (A_\mu^{(1)} + A_\mu^{(2)}) \right\} (1-2x) - \frac{2 A_\mu^{(1)} A_\mu^{(2)} \{1 - \delta(x)\}}{t} \right] \quad , \end{aligned} \quad (\text{A.12})$$

with $\epsilon_{\mu 4} = -1$ if $\mu = 4$ and 1 otherwise. The dependence on the vector field can then be expanded again, to recover the specific quadratic terms discussed in more detail below.

1. The A_4 - A_4 part

Consider first the terms involving the A_4 field. Since we are using the gauge condition $\partial_4 A_4 = 0$ to simplify the actual calculation, Eq. (A.12) changes in that there is no p_4 -dependence when the spatial components $A_k = 0$. Instead of a sum over Matsubara frequencies the integration over z_4 is left over; in other words, we do a three-dimensional Fourier transform of A_4 instead of Eq. (A.1). By expanding the A_4 -dependent exponential,

$$e^{\left[n\beta(A_4^{(1)}+A_4^{(2)})/t - \frac{2}{t}A_4^{(1)}A_4^{(2)}(1-\delta(x))/t \right]} \Big|_{\text{linear in each } A_4^{(i)}} = \left(\frac{n^2\beta^2}{t^2} - \frac{2(1-\delta(x))}{t} \right) A_4^{(1)}A_4^{(2)}, \quad (\text{A.13})$$

and using the following integral representation of the Bessel function of the second kind,

$$K_\nu(xz) = \frac{z^\nu}{2} \int_0^\infty \frac{dt}{t^{\nu+1}} \exp \left[-\frac{x}{2} \left(t + \frac{z^2}{t} \right) \right], \quad (\text{A.14})$$

we obtain Eq. (18).

2. The A_k - A_j part

Turning now to the spatial parts of the vector field, we first use Eq. (A.12) again and put $A_4 = 0$. We then use the expansion

$$e^{\left[ip_k(A_k^{(1)}+A_k^{(2)})(1-2x) - 2\vec{A}^{(1)} \cdot \vec{A}^{(2)}(1-\delta(x))/t \right]} \Big|_{\text{linear in each } A_k^{(i)}} = - \left[\delta_{kj} \frac{2(1-\delta(x))}{t} + p_k p_j (1-2x)^2 \right] A_k^{(1)} A_j^{(2)}, \quad (\text{A.15})$$

to obtain Eq. (21). Using Eq. (A.14) and performing a partial integration in x , we then get Eq. (23). To find the leading T -dependence in the effective action, only the sums over the square-root terms in Eq. (24) need to be kept, and by changing $m \rightarrow -(m+1)$ in the second square-root we have,

$$L_{\text{soft}(2)}^{(2)} \Big|_{T>0} = \left(\frac{e^2}{8\pi} \right) \int_0^1 dx (1-2x) T \sum_{m=-\infty}^\infty \left[\frac{p_k p_j A_k^{(1)} A_j^{(2)} (1-2x)}{\sqrt{-p^2 x^2 + (P^2 - 2k_4 p_4)x + k_4^2}} + \right. \\ \left. + 2 \int_0^x d\sigma \frac{\vec{A}^{(1)} \cdot \vec{A}^{(2)}}{\sqrt{P^2 x(1-x) + [k_4 - \sigma p_4]^2}} \right], \quad (\text{A.16})$$

where $k_4 = (2m+1)\pi T$. By integrating over σ and performing a partial integration over x in the second term inside the square brackets, Eq. (A.16) becomes,

$$L_{\text{soft}(2)}^{(2)} \Big|_{T>0} = \left(\frac{e^2}{8\pi} \right) T \sum_{m=-\infty}^\infty \left[-4\vec{A}^{(1)} \cdot \vec{A}^{(2)} \left\{ \sqrt{(k_4 - p_4)^2} \right\} + \right. \\ \left. + \int_0^1 dx \left(\frac{p_k p_j A_k^{(1)} A_j^{(2)} (1-2x)^2 + 2x \vec{A}^{(1)} \cdot \vec{A}^{(2)} (P^2 - 2p^2 x - 2k_4 p_4)}{\sqrt{-p^2 x^2 + (P^2 - 2k_4 p_4)x + k_4^2}} \right) \right]. \quad (\text{A.17})$$

To proceed from the above equation, we use that

$$\int_0^\infty \frac{d\theta \theta^2}{(\theta^2 + a)^2} = \frac{\pi}{4\sqrt{a}}. \quad (\text{A.18})$$

Since we are discarding all terms but the leading T dependence, we can in fact neglect the term $p_k p_j A_k^{(1)} A_j^{(2)}$ in the numerator $p_k p_j A_k^{(1)} A_j^{(2)} (1-2x)^2$. This can be shown by using Eq. (A.18) and performing the x -integration. We can then write Eq. (A.17) as,

$$L_{\text{soft}(2)}^{(2)}|_{T>0} = \left(\frac{e^2}{2\pi^2} \right) T \sum_{m=-\infty}^{\infty} \left[\int_0^{\infty} d\theta \theta^2 \left\{ \vec{A}^{(1)} \cdot \vec{A}^{(2)} \left(-\delta(\theta-1) \pi \sqrt{(k_4-p_4)^2} - \right. \right. \right. \\ \left. \left. \left. - \frac{2}{\theta^2 + (k_4-p_4)^2} \right) + A_k^{(1)} A_j^{(2)} \frac{\partial^2}{\partial p_k \partial p_j} \int_0^1 \frac{dx}{x(1-x)} \log \left[P^2 x(1-x) + \theta^2 + (k_4 - xp_4)^2 \right] \right\} \right] . \quad (\text{A.19})$$

In the first and second term, we write the sum over m as a contour integral and keep only the finite T part. For the first term the contribution to the contour integral comes from the cut of the square-root, whereas for the second the contour can be closed and the contribution evaluated by the residue theorem. Remarkably, the two contributions cancel each other and we are left with the last term in Eq. (A.19).

APPENDIX B: THE HIGHER ORDER CONTRIBUTIONS

In going from the expressions for $I_{N(i)}^{(k)}$ in Eqs. (37)-(40) to the results in Eqs. (42)-(44), we follow the general method described in [12]. To be specific, consider the the sum over winding modes in Eq. (36) coming from $I_{N(1)}^{(k)}$, Eq. (37),

$$\sum_{n=1}^{\infty} (-1)^n n^{N-2} K_{2-N+k}(n\beta\tilde{p}) . \quad (\text{B.1})$$

Using a different integral representation for the Bessel function,

$$K_{\nu}(x) = \int_0^{\infty} dt \cosh(\nu t) e^{-x \cosh(t)} , \quad (\text{B.2})$$

we have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n n^{N-2} K_{2-N+k}(n\beta\tilde{p}) &= \int_0^{\infty} dt \cosh[(2-N+k)t] \sum_{n=1}^{\infty} (-1)^n n^{N-2} e^{-n\beta\tilde{p} \cosh(t)} \\ &= \int_0^{\infty} dt \frac{\cosh[(2-N+k)t]}{[\tilde{p} \cosh(t)]^{N-2}} \left(\frac{\partial}{\partial \beta} \right)^{N-2} \sum_{n=1}^{\infty} (-1)^n e^{-n\beta\tilde{p} \cosh(t)} \\ &= \frac{-1}{\tilde{p}^{N-2}} \left(\frac{\partial}{\partial \beta} \right)^{N-2} \int_0^{\infty} dt \frac{\cosh[(2-N+k)t]}{[\cosh(t)]^{N-2}} \left(\frac{1}{e^{\beta\tilde{p} \cosh(t)} + 1} \right) . \end{aligned} \quad (\text{B.3})$$

Now, with the relation

$$\begin{aligned} \cosh[(2-N+k)t] &= \cosh[(N-2-k)t] = 2^{|N-2-k|-1} [\cosh(t)]^{|N-2-k|} + |N-2-k| \times \\ &\sum_{m=1}^M \frac{(-1)^m}{m} \binom{|N-2-k|-m-1}{m-1} 2^{|N-2-k|-2m-1} [\cosh(t)]^{|N-2-k|-2m} , \end{aligned} \quad (\text{B.4})$$

where the sum over m runs up to the largest integer $M \leq |N-2-k|/2$, we have for the different values of k ,

$$\begin{aligned} & \frac{-1}{\tilde{p}^{N-2}} \left(\frac{\partial}{\partial \beta} \right)^{N-2} \int_0^\infty \frac{dt}{e^{\beta \tilde{p} \cosh(t)} + 1} \left[\delta_{kN} \left\{ \frac{2[\cosh(t)]^2 - 1}{[\cosh(t)]^{N-2}} \right\} + \delta_{k(N-2)} \left\{ \frac{1}{[\cosh(t)]^{N-2}} \right\} + \right. \\ & \left. \sum_{j=0}^{N-4} 2^{N-k-3} \delta_{kj} \left\{ \frac{1}{[\cosh(t)]^j} + \sum_{m=1}^M \frac{a_m^{(1)}}{[\cosh(t)]^{j+2m}} \right\} \right] . \end{aligned} \quad (\text{B.5})$$

Here $a_m^{(1)}$ captures the numerical factors in Eq. (B.4).

To proceed, we now use the Mellin transform,

$$g(s) = \int_0^\infty dx f(x) x^{s-1} \quad (\text{B.6})$$

$$f(x) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} ds g(s) x^{-s} , \quad (\text{B.7})$$

to write

$$(e^x + 1)^{-1} = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} ds \Gamma(s) (1 - 2^{1-s}) \zeta(s) x^{-s} . \quad (\text{B.8})$$

Using this equation we can perform the t -integration (the integral over powers of the hyperbolic cosine function gives ratios of Γ -functions [35]), and by taking the $(N-2)$ derivatives with respect to β and inserting the prefactors of T and p , we finally obtain Eq. (42).

In scalar QED, the situation is almost identical, with a few notable differences. Firstly, the overall result has to be multiplied by $(-1/2)$, to correct for the Fermi statistics and the number of degrees of freedom. Secondly, the sum over the winding modes in Eq. (B.3) has no factor $(-1)^n$, so instead of $(-1/[e^x + 1])$ one has a Bose-Einstein distribution, $(1/[e^x - 1])$. The function $g(s)$ in the Mellin transform then changes to $\Gamma(s)\zeta(s)$, and the vertical contour in the complex s -plane must now be along $\text{Re}(s) = 1 + \epsilon$.

REFERENCES

- [1] J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
- [2] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
- [3] R. D. Pisarski, Phys. Rev. Lett. **63**, 1129 (1989).
- [4] E. Braaten and R. D. Pisarski, Phys. Rev. Lett. **64**, 1338 (1990); Nucl. Phys. **B337**, 569 (1990); *ibid* **B339**, 310 (1990).
- [5] J. Frenkel and J. C. Taylor, Nucl. Phys. **334**, 199 (1990).
- [6] J. C. Taylor and S. M. H. Wong, Nucl. Phys. **B346**, 115 (1990).
- [7] E. Braaten and R. D. Pisarski, Phys. Rev. **D45**, R1827 (1992).
- [8] J.-P. Blaizot and E. Iancu, Nucl. Phys. **B390**, 589 (1993); Phys. Rev. Lett. **70**, 3376 (1993); Nucl. Phys. **B417**, 608 (1994).
- [9] P. F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, Phys. Rev. Lett. **72**, 3461 (1994); Phys. Rev. **D50**, 4209 (1994).
- [10] P. Elmfors, T. H. Hansson and I. Zahed, Phys. Rev. **D59**, 045018 (1999).
- [11] C. Schubert, hep-th/**0101036**, to appear in Phys. Rept.
- [12] D. G. C. McKeon and A. Rebhan, Phys. Rev. **D47**, 5487 (1993).
- [13] D. G. C. McKeon and A. Rebhan, Phys. Rev. **D49**, 1047 (1994).
- [14] H.-T. Sato, J. Math. Phys. **40**, 6407 (1999).
- [15] I. A. Shovkovy, Phys. Lett. **B441**, 313 (1998).
- [16] F. T. Brandt and D. G. C. McKeon, Phys. Rev. **D54**, 6435 (1996).
- [17] H. Gies, Phys. Rev. **D61**, 085021 (2000).
- [18] D. L. Burke *et al.*, Phys. Rev. Lett. **79**, 1626 (1997).
- [19] A. K. Harding, Phys. Rept. **206**, 328 (1991); M. G. Baring and A. K. Harding, *Proceedings of the 1995 La Jolla Workshop* (AIP, New York, 1995).
- [20] P. Elmfors and B.-S. Skagerstam, Phys. Lett. **B427**, 197 (1998).
- [21] F. T. Brandt and J. Frenkel, Phys. Rev. Lett. **74**, 1705 (1995).
- [22] F. T. Brandt, J. Frenkel and J. C. Taylor, Phys. Rev. **D50**, 4110 (1994).
- [23] S. R. DeGroot, W. Van Leeuwen and Ch. G. Van Weert, *Relativistic Kinetic Theory* (North Holland, Amsterdam, 1980); W. Israel, Ann. Phys. **100**, 310 (1976).
- [24] R. D. Pisarski, hep-ph/**9710370**.
- [25] J. Jalilian-Marian, S. Jeon, R. Venugopalan and J. Wirstam, Phys. Rev. **D62**, 045020 (2000).
- [26] J. Jalilian-Marian, S. Jeon and R. Venugopalan, Phys. Rev. **D63**, 036004 (2001).
- [27] N. P. Landsman, Nucl. Phys. **B322**, 498 (1989).
- [28] M. J. Strassler, Nucl. Phys. **B385**, 145 (1992).
- [29] Z. Bern and D. A. Kosower, Nucl. Phys. **B379**, 451 (1992).
- [30] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, Chur, Switzerland, 1987).
- [31] L. Brink, P. Di Vecchia and P. Howe, Nucl. Phys. **B118**, 76 (1977).
- [32] E. D'Hoker and D. G. Gagne, Nucl. Phys. **B467**, 272 (1996).
- [33] L. F. Abbott, Acta Phys. Polon. **B13**, 33 (1982).
- [34] E. S. Fradkin, Proc. Lebedev Inst. **29**, 6 (1965).

- [35] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1980).
- [36] A. V. Smilga, Phys. Rept. **291**, 1 (1997).
- [37] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
- [38] S. Nadkarni, Phys. Rev. **D27**, 917 (1983); *ibid* **D38**, 3287 (1988).
- [39] J. O. Andersen, E. Braaten and M. Strickland, Phys. Rev. Lett. **83**, 2139 (1999); Phys. Rev. **D61**, 074016 (2000).
- [40] J. P. Blaizot, E. Iancu and A. Rebhan, Phys. Rev. Lett. **83**, 2906 (1999).
- [41] E. Braaten and A. Nieto, Phys. Rev. Lett. **76**, 1417 (1996); Phys. Rev. **D53**, 3421 (1996).
- [42] K. Kajantie, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. **B407**, 356 (1993); K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. **B458**, 90 (1996).
- [43] R. D. Pisarski, Phys. Rev. **D62**, 111501 (2000).
- [44] R. Venugopalan and J. Wirstam, in preparation.
- [45] A. Dumitru and R. D. Pisarski, hep-ph/**0010083**.